RATLIFF-RUSH CLOSURE OF IDEALS IN INTEGRAL DOMAINS

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Abstract. This paper studies the Ratliff-Rush closure of ideals in integral domains. By definition, the Ratliff-Rush closure of an ideal I of a domain R is the ideal given by $\tilde{I} := \bigcup(I^{n+1} :_R I^n)$ and an ideal I is said to be a Ratliff-Rush ideal if $\tilde{I} = I$. We completely characterize integrally closed domains in which every ideal is a Ratliff-Rush ideal and we give a complete description of the Ratliff-Rush closure of an ideal in a valuation domain.

1. Introduction

Let R be a commutative ring with identity and I a regular ideal of R, that is, I contains a nonzero divisor. The ideals of the form $(I^{n+1} :_R I^n) := \{x \in R | xI^n \subseteq I^{n+1}\}$ increase with n. In the case where R is a Noetherian ring, the union of this family is an interesting ideal, first studied by Ratliff and Rush in [22]. In [12], W. Heinzer, D. Lantz and K. Shah called the ideal $\tilde{I} := \bigcup(I^{n+1} :_R I^n)$ the Ratliff-Rush closure of I, or the Ratliff-Rush ideal associated with I. An ideal I is said to be a Ratilff-Rush ideal, or Ratliff-Rush closed, if $I = \tilde{I}$. Among the interesting facts of this ideal is that, for any regular ideal I in a Noetherian ring R, there exists a positive integer n such that for all $k \geq n$, $I^k = (\tilde{I})^k$, that is, all sufficiently high powers of a regular ideal are Ratliff-Rush ideals, and a regular ideal is always a reduction of its Ratliff-Rush closure in the sense of Northcoot-Rees (see [17]), that is, $I(\tilde{I})^n = (\tilde{I})^{n+1}$ for some positive integer n. Also the ideal $\tilde{I}$ is always between I and the integral closure I' of I, that is, $I \subseteq \tilde{I} \subseteq I'$, where $I' := \{x \in R | x$ satisfies an equation of the form $x^k + a_1x^{k-1} + \cdots + a_k = 0$, where $a_i \in I^i$ for each $i \in \{1, \ldots, k\}\}$. Therefore, integrally closed ideals, i. e., ideals such that $I = I'$, are Ratliff-Rush ideals. Since then, many investigations of the Ratliff-Rush closure of ideals in a Noetherian ring have been carried out, for instance, see [11], [12], [16], [23] etc. The purpose of this paper is to extend the notion of Ratliff-Rush closure of ideals to an arbitrary integral domain and examine ring-theoretic properties of this kind of closure. In the second section, we give an answer to a question raised by B. Olberding [21] about the classification of integral domains for which every

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ideal is a Ratliff-Rush ideal in the context of integrally closed domains. This lead us to give a new characterizations of Prüfer and strongly discrete Prüfer domains. Specifically, we prove that "a domain R is a Prüfer (respectively strongly discrete Prüfer) domain if and only if R is integrally closed and each nonzero finitely generated (respectively each nonzero) ideal of R is a Ratliff-Rush ideal" (Theorem 2.6). It turns that a Ratliff-Rush domain (i. e., domain such that each nonzero ideal is a Ratliff-Rush ideal) is a quasi-Prüfer domain, that is, its integral closure is a Prüfer domain. As an immediate consequence, we recover Heinzer-Lantz-Shah's results for Noetherian domains (Corollary 2.8). The third section deals with valuation domains. Here, we give a complete description of the Ratliff-Rush closure of a nonzero ideal in a valuation domain (Proposition 3.2), and we state necessary and sufficient condition under which the Ratliff-Rush closure preserves inclusion (Proposition 3.3). We also extend the Ratliff-Rush closure to arbitrary nonzero fractional ideals of a domain R, and we investigate its link to the notions of star operations. We prove that "for a valuation domain V, the Ratliff-Rush closure is a star operation if and only if every nonzero nonmaximal prime ideal of V is not idempotent, and in this case it coincides with the v-closure" (Theorem 3.5).

Throughout, R denotes an integral domain, $qf(R)$ its quotient field, and R' and $\overline{R}$ its integral closure and complete integral closure respectively. For a nonzero (fractional) ideal I of R, the inverse of I is given by $I^{-1} = (R : I) := \{x \in qf(R) | xI \subseteq R\}$. The v-closure and t-closure are defined respectively by $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup J_v$ where J ranges over the set of f. g. subideals of I. We say that I is divisorial (or a v-ideal) if $I = I_v$, and a t-ideal if $I = I_t$. Unreferenced material is standard as in [10] or [15].

2. Ratliff-Rush ideals in an integral domain

Let R be an integral domain. A nonzero ideal I of R is L-stable (here L stands for Lipman) if $R^I := \bigcup (I^n : I^n) = (I : I)$. The ideal I is stable (or Sally-Vasconcelos stable) if I is invertible in its endomorphisms ring $(I : I)$ ([24]). A domain R is L-stable (respectively stable) if every nonzero ideal of R is L-stable (respectively stable). We recall that a stable domain is L-stable [1, Lemma 2.1], and for recent developments on stability (in settings different than originally considered), we refer the reader to [1, 18, 19, 20]. We start this section with the following definition which extend the notion of Ratliff-Rush closure to nonzero integral ideals in an arbitrary integral domain.

Definition 2.1. Let R be an integral domain and I a nonzero integral ideal of R. The Ratliff-Rush closure of I is the (integral) ideal of R given by
$\tilde{I} = \bigcup (I^{n+1} :_R I^n)$. An integral ideal I of R is said to be a Ratliff-Rush ideal, or Ratliff-Rush closed, if $I = \tilde{I}$, and R is said to be a Ratliff-Rush domain if each nonzero integral ideal of R is a Ratliff-Rush ideal.

The following useful lemma treats the Ratliff-Rush closure of some particular classes of ideals.

Lemma 2.2. *Let R be an integral domain. Then:*
1-All stable (and thus all invertible) ideals are Ratliff-Rush.
2-If I is a nonzero idempotent ideal of R, then $\tilde{I} = R$.

Proof. 1) Let I be a stable ideal of R and set $T = (I : I)$. Then $I(T : I) = T$. Now, let $x \in \tilde{I}$. Then $x \in R$ and $xI^s \subseteq I^{s+1}$ for some positive integer s. Composing the two sides with $(T : I)$ and using the fact that $I(T : I) = T$, we obtain $xI^{s-1} \subseteq I^s$. Iterating this process, we get $xT \subseteq I$. Hence $x \in I$ and therefore $I = \tilde{I}$, as desired.
2) Let I be a nonzero idempotent ideal of R. Then for each n, $I^n = I$. So $(I^{n+1} :_R I^n) = (I :_R I) = (I : I) \cap R = R$. Hence $\tilde{I} = R$.

□

The next proposition relates the Ratliff-Rush closure to the L-stability.

Proposition 2.3. *Let R be an integral domain. If R is a Ratliff-Rush domain, then R is L-stable.*

Proof. Assume that R is a Ratliff-Rush domain. Let I be a nonzero (integral) ideal of R and let $x \in R^I$. Then there exists a positive integer n such that $xI^n \subseteq I^n$. Let $0 \neq d \in R$ such that $dx \in R$. Then $xI^{n+1} \subseteq I^{n+1}$ implies that $dxI(dI)^n = d^{n+1}xI^{n+1} \subseteq d^{n+1}I^{n+1} = (dI)^{n+1}$. Hence $dxI \subseteq ((dI)^{n+1} : (dI)^n)$. Since $dxI \subseteq R$, then $dxI \subseteq \widetilde{(dI)} = dI$ (since R is Ratliff-Rush) and so $xI \subseteq I$. Hence $x \in (I : I)$ and therefore $R^I = (I : I)$. So I is L-stable and therefore R is L-stable, as desired. □

It's easy to see that for a finitely generated ideal I of a domain R, in particular if R is Noetherian, $\tilde{I} \subseteq I'$. However, this is not the case for an arbitrary ideal of an integral domain. Indeed, let V be a valuation domain with maximal ideal M such that $M^2 = M$, $0 \neq a \in M$ and set $I = aM$. It is easy to see that $\tilde{I} = a(M : M) \cap V = aV$ and $I = I'$ (since all ideals of a Prüfer domains are integrally closed). The next theorem establishes a connection between stable domains, Ratliff-Rush domains and domains for which $\tilde{I} \subseteq I'$ for all ideals I. For this, we need the following crucial lemma.

Lemma 2.4. *Let R be an integral domain. If $\tilde{I} = I$ for every finitely generated ideal I of R, then R' is a Prüfer domain.*

Proof. Let N be a maximal ideal of R'. To show that R'_N is a valuation domain, let $x = \frac{a}{b} \in qf(R)$, where $a, b \in R\backslash\{0\}$. Let J be the ideal (a^4, a^3b, ab^3, b^4) of R. Then $a^2b^2J = (a^6b^2, a^5b^3, a^3b^5, a^2b^6) \subseteq J^2 = (a^8, a^7b, a^5b^3, a^4b^4, a^6b^2, a^3b^5, a^2b^6, ab^7, b^8)$. So $a^2b^2 \in (J^2 :_R J) \subseteq \tilde{J} = J$. Thus $a^2b^2 = g_1a^4 + g_2a^3b + g_3ab^3 + g_4b^4$ for some g_1, g_2, g_3 and g_4 in R. Dividing by b^4, we get $0 = g_1x^4 + g_2x^3 - x^2 + g_3x + g_4$. By the u, u^{-1} theorem ([15, Theorem 67]), we get that either $x \in R'_N$ or $x^{-1} \in R'_N$, as desired. □

Theorem 2.5. *Let R be an integral domain. Consider the following.*
(1) *R is stable.*
(2) *R is Ratliff-Rush.*
(3) *$\tilde{I} \subseteq I'$ for each nonzero ideal I of R.*
(4) *R has no nonzero idempotent prime ideals.*
Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). *Moreover, If R is a semilocal Prüfer domain, then* (4) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) by Lemma 2.2.
(2) $\Longrightarrow$ (3) is clear.
For (3) $\Longrightarrow$ (4), assume that P is a nonzero idempotent prime ideal of R. Then if $I = aP$ with $0 \neq a \in P$, then for all $n \geq 1$, $(I^{n+1} :_R I^n) = (I^{n+1} : I^n) \cap R = (a^{n+1}P : a^nP) \cap R = a(P : P) \cap R = a(P : P)$ (since $a(P : P) \subseteq P(P : P) = P \subseteq R$). So $a \in a(P : P) = \tilde{I}$. Suppose $a \in I' = (aP)'$. Then $a^k + c_1a^{k-1} + \cdots + c_k = 0$, where $c_i = a^ib_i \in I^i = a^iP$ for each $i \in \{1, \ldots, k\}\}$. So $a^k + b_1a^k + b_2a^k + \cdots + b_ka^k = 0$ with $b_i \in P$. Thus $a^{k+1}(1 + b) = 0$ with $b \in P$, a contradiction.
(4) $\Longleftrightarrow$ (1) if R is a semilocal Prüfer domain by [1, Theorem 2.10]. □

We are now ready to announce the main theorem of this section. It gives a classification of the integral domains for which every ideal is a Ratliff-Rush ideal in the context of integrally closed domains and states a new characterization of Prüfer and strongly discrete Prüfer domains. Recall that a Prüfer domain is said to be strongly discrete if $P \neq P^2$ for each nonzero prime ideal P of R.

Theorem 2.6. *Let R be an integrally closed domain. The following statements are equivalent.*
(1) *$\tilde{I} = I$ for every finitely generated (respectively every) nonzero ideal I of R.*
(2) *R is Prüfer (respectively strongly discrete Prüfer).*

Proof. (1) $\Longrightarrow$ (2) By Lemma 2.4, R is a Prüfer domain. Moreover, if each ideal is a Ratliff-Rush ideal, by Theorem 2.5, R is strongly discrete.

(2) $\Longrightarrow$ (1). Let R be a Prüfer domain. Then every finitely generated ideal is invertible and therefore a Ratliff-Rush ideal by Lemma 2.2. Assume that R is a strongly discrete Prüfer domain. Let I be a nonzero ideal of R and let $x \in \tilde{I}$.

Then $x \in R$ and $xI^s \subseteq I^{s+1}$ for some positive integer s. Let M be a maximal ideal of R. If $I \nsubseteq M$, then $x \in R \subseteq R_M = IR_M$. Assume that $I \subseteq M$. Since $x \in R_M$ and $xI^sR_M \subseteq I^{s+1}R_M$, then $x \in \widetilde{IR_M}$. Since R is strongly discrete, then R_M is a strongly discrete valuation domain. By Theorem 2.5, $\widetilde{IR_M} = IR_M$. Hence $x \in IR_M$. So $x \in \bigcap\{IR_M/M \in Max(R)\} = I$. Hence $I = \tilde{I}$, as desired. $\square$

The following example shows that the above Theorem is not true if R is not integrally closed.

Example 2.7. Let $\mathbb{Q}$ be the field of rational numbers, X an indeterminate over $\mathbb{Q}$ and $V = \mathbb{Q}(\sqrt{2})[[X]] = \mathbb{Q}(\sqrt{2}) + M$. Set $R = \mathbb{Q} + M$. Then R is stable. Indeed, Let I be a nonzero (integral) ideal of R. Since R is local with maximal ideal M, then $I \subseteq M$. If I is an ideal of V, then $I = cV$ for some $c \in I$. If I is not an ideal of V, then $I = m(W + M)$, where $\mathbb{Q} \subseteq W \subsetneqq \mathbb{Q}(\sqrt{2})$ is a $\mathbb{Q}$-vector space. Since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, then $\mathbb{Q} = W$ and so $I = cR$. Therefore R is stable and then Ratliff-Rush by Theorem 2.5. However, R is not a Prüfer domain ([4, Theorem 2.1]).

Our next corollary recovers Heinzer-Lantz-Shah's results for Noetherian domains.

Corollary 2.8. *(cf.* [12, Proposition 3.1 and Theorem 3.9] *Let R be a Noetherian domain. Then R is a Ratliff-Rush domain if and only if R is stable.*

Proof. Since R is Noetherian, then $R' = \bar{R}$ is a Krull domain. By Lemma 2.4, R' is a Prüfer domain. Hence R' is a Dedekind domain and therefore $dimR = dimR' = 1$. By Proposition 2.3, R is L-stable and therefore stable by [1, Proposition 2.4]. $\square$

We recall that a domain R is said to be strong Mori if R satisfies the ascending chain conditions on w-ideals [7]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. The next corollary shows that the Ratliff-Rush property forces a strong Mori domain to be Noetherian.

Corollary 2.9. *Let R be a strong Mori domain. If R is a Ratliff-Rush domain, then R is Noetherian.*

Proof. By Lemma 2.4, R' is a Prüfer domain. Hence every maximal ideal of R is divisorial ([5, Corollary 2.5] and [6, Theorem 2.6]). Now, let M be a maximal ideal of R. Since $M = M_v$, then R_M is Noetherian ([7, Theorem 3.9]). Hence $R'_M = (R_M)' = \overline{R_M}$ is a Krull domain. But since R' is Prüfer, then so is R'_M. Hence R'_M is Dedekind and so $htM = dimR_M = dimR'_M = 1$. Then $dimR = 1$ and therefore R is Noetherian ([7, Corollary 3.10]). $\square$

Recall that R is seminormal if for each $x \in qf(R)$, $x^2, x^3 \in R$ implies that $x \in R$. Our next corollary states some conditions under which a Ratliff-Rush Mori domain has dimension one.

Corollary 2.10. *Let R be a Mori domain such that either $(R : \overline{R}) \neq 0$ or R is seminormal. If R is a Ratliff-Rush domain, then $dim R = 1$.*

Proof. Assume that R is a Ratliff-Rush domain. By Lemma 2.4, R' is a Prüfer domain.
(1) If $(R : \overline{R}) \neq (0)$, then $\overline{R}$ is a Krull domain ([2, Corollary 18]). Since $R' \subseteq \overline{R}$, then $\overline{R}$ is a Prüfer domain, and therefore Dedekind. Hence $\dim(\overline{R}) = 1$. By [3, Corollary 3.4], $\dim(R) = 1$, as desired.
(2) Assume that R is seminormal. If $\dim(R) \geq 2$, then R has a maximal ideal M such that $htM \geq 2$. Set $B = (MR_M)^{-1} = (MR_M : MR_M)$. Since R_M is a local Mori domain which is seminormal and $htMR_M = htM \geq 2$, then B contains a nondivisorial maximal ideal N contracting to MR_M ([3, Lemma 2.5]). Since R' is a Prüfer domain (Lemma 2.4) and combining [5, Corollary 2.5]) and [6, Theorem 2.6], we get that every maximal ideal of B is a t-ideal and so a v-ideal since B is Mori, which is absurd. Hence $\dim(R) = 1$, as desired. □

3. Ratliff-Rush ideals in a Valuation domain

It's well-known that the maximal ideal M of a valuation domain V is either principal or idempotent, any nonzero prime ideal P of V is a divided prime ideal, that is, $PV_P = P$, and any idempotent ideal is a prime ideal. Also we recall that a valuation domain is a TP domain, that is, for each nonzero ideal I of V, either $II^{-1} = V$ or $II^{-1} = Q$ is a prime ideal of V ([8, Proposition 2.1]), and for each positive integer n, $I^nI^{-n} = II^{-1}$ ([13, Remark 2.13(b)]). We will often use this facts without explicit mention. Finally V is strongly discrete if it has no nonzero idempotent prime ideal ([9, chapter 5.3]).

Lemma 3.1. *Let V be a valuation domain, I a nonzero ideal of V and assume that $\tilde{I} \neq V$. Then $(I : I) \subseteq (\tilde{I} : \tilde{I})$.*

Proof. Let I be a nonzero ideal of V and assume that $\tilde{I} \neq V$. If $II^{-1} = V$, then $I = \tilde{I}$ by Lemma 2.2 and therefore $(I : I) = (\tilde{I} : \tilde{I})$. Assume that $II^{-1} = Q$ is a prime ideal of V. Since V is a valuation domain, then V is L-stable. So $(I : I) = (I^n : I^n)$ for each positive integer n. Let $x \in (I : I)$ and $z \in \tilde{I}$. Then $z \in V$ and $zI^r \subseteq I^{r+1}$ for some positive integer r. Since $(I : I) = (I^{r+1} : I^{r+1})$, then $xzI^r \subseteq xI^{r+1} \subseteq I^{r+1}$. Hence $xz \in (I^{r+1} : I^r)$. To show that $xz \in \tilde{I}$, it suffices to prove that $xz \in V$. Suppose that $xz \notin V$. Then $(xz)^{-1} \in V$. Since $z \in \tilde{I}$, then $x^{-1} = (xz)^{-1}z \in \tilde{I}$. So $x^{-1} \in V$ and $x^{-1}I^s \subseteq I^{s+1}$ for some positive integer s. Hence $I^s \subseteq xI^{s+1} \subseteq I^{s+1}$ (since $(I : I) = (I^{s+1} : I^{s+1})$) and therefore $I^s = I^{s+1}$. Hence $I^s = I^{2s}$ and therefore $I = P$ is an idempotent prime ideal of

V. By Lemma 2.2, $\tilde{I} = \tilde{P} = V$, which is absurd. Hence $xz \in V$. So $xz \in \tilde{I}$ and then $x\tilde{I} \subseteq \tilde{I}$. Hence $x \in (\tilde{I} : \tilde{I})$ and therefore $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I})$.

□

The next proposition describes the Ratliff-Rush closure of a nonzero integral ideal in a valuation domain.

Proposition 3.2. *Let I be a nonzero integral ideal of a valuation domain V. Then:*
(1) *$\tilde{I} = V$ if and only if I is an idempotent prime ideal of V.*
(2) *Assume that $\tilde{I} \subsetneq V$. Then either $\tilde{I} = I$, or $\tilde{I} = (IQ :_V Q)$ for some nonzero prime ideal Q of V.*

Proof. (1) If I is an idempotent prime ideal of V, by Lemma 2.2, $\tilde{I} = V$. Conversely, assume that $\tilde{I} = V$. Then there exists a positive integer n such that $I^n \subseteq I^{n+1}$. Hence $I^n = I^{n+1}$. By induction, $(I^n)^2 = I^n$. So I^n is an idempotent ideal of V. Hence $I^n = P$ is a prime ideal of V. Then $I \subseteq P \subseteq I$ and therefore $I = P$, as desired.

(2) Assume that $\tilde{I} \subsetneq V$. If $II^{-1} = V$, then $I = \tilde{I}$ by Lemma 2.2. Assume that $II^{-1} = Q \subsetneq V$ is a prime ideal. Then $(I : I) = V_Q$ and for each positive integer n, $I^n I^{-n} = Q$ since V is a TP domain. Let $x \in \tilde{I}$. Then $x \in V$ and $xI^n \subseteq I^{n+1}$ for some positive integer n. So $xQ = xI^n I^{-n} \subseteq xI^{n+1} I^{-n} = IQ$. Hence $x \in (IQ :_V Q)$ and therefore $\tilde{I} \subseteq (IQ :_V Q)$. Now, assume that $I \subsetneq \tilde{I} \subsetneq V$. To complete the proof, we will show that $\tilde{I} = (IQ :_V Q)$. Since $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I})$ (Lemma 3.1), then $\tilde{I}$ is an ideal of V_Q. Suppose that $\tilde{I} \subsetneq (IQ :_V Q)$. Let $x \in (IQ :_V Q) \setminus \tilde{I}$. Since V is a valuation domain, then $\tilde{I} \subsetneq xV$. So $x^{-1}\tilde{I} \subsetneq V \subseteq V_Q$. Hence $x^{-1}\tilde{I}$ is a proper ideal of V_Q. So $x^{-1}\tilde{I} \subseteq Q$ ($Q = QV_Q$ is the maximal ideal of V_Q). Hence $\tilde{I} \subseteq xQ \subseteq IQ \subseteq I \subsetneq \tilde{I}$, a contradiction. It follows that $\tilde{I} = (IQ :_V Q)$, as desired. □

Our next proposition shows that the Ratliff-Rush closure of an ideal I in a valuation domain is itself a Ratliff-Rush ideal, and gives necessary and sufficient condition for preserving the Ratliff-Rush closure under inclusion.

Proposition 3.3. *Let I be a nonzero ideal of a valuation domain V. Then*
1) $\tilde{\tilde{I}} = \tilde{I}$.
2) $\tilde{I} \subseteq \tilde{J}$ for every ideals $I \subseteq J$ if and only each nonzero nonmaximal prime ideal of V in not idempotent.

Proof. 1) If $I = \tilde{I}$ or $\tilde{I} = V$, then clearly $\tilde{\tilde{I}} = \tilde{I}$. Assume that $I \subsetneq \tilde{I} \subsetneq V$. By Proposition 3.2, $\tilde{I} = (IQ :_V Q)$ where $Q = II^{-1}$ is a prime ideal of V (note that $II^{-1} \subsetneq V$, otherwise $I = \tilde{I}$, by Lemma 2.2). For simplicity, we set $J = \tilde{I}$. Our

aim is to prove that $J = \tilde{J}$. If $JJ^{-1} = V$, then $J = \tilde{J}$ by Lemma 2.2. Assume that $JJ^{-1} \subsetneq V$. By Lemma 3.1, $V_Q = (I : I) \subseteq (\tilde{I} : \tilde{I}) = (J : J) = V_P$, where $P = JJ^{-1}$. So $P \subseteq Q$. Let $x \in \tilde{J}$. Then $x \in V$ and $xJ^n \subseteq J^{n+1}$ for some positive integer n. Composing the two sides with J^{-n} and using the fact that $P = JJ^{-1} = J^nJ^{-n}$, we obtain $xP \subseteq JP$. Hence $\tilde{J}P \subseteq JP \subseteq JQ = \tilde{I}Q = IQ$. Now, if $P \subsetneq Q$, then let $a \in Q \setminus P$. Since V is a valuation domain, then $P \subsetneq aV$. So $a^{-1}P \subsetneq V$. Hence $a^{-1} \in (V : P) = (P : P) = V_P = (J : J)$ ([14]). So $a^{-1}J \subseteq J$. Then $J \subseteq aJ \subseteq QJ = QI \subseteq I \subsetneq J$, a contradiction. Hence $P = Q$. So $\tilde{J}P = \tilde{J}Q = JQ = IQ$. Hence $\tilde{J} \subseteq (IQ :_V : Q) = \tilde{I} = J$, as desired.

2) Assume that $\tilde{I} \subseteq \tilde{J}$ for every ideals $I \subseteq J$. Suppose that there is a nonzero nonmaximal prime ideal P of V such that $P^2 = P$. Let $a \in M \setminus P$, where M is the maximal ideal of V. Since V is a valuation domain, then $P \subsetneq aV = I$. By Lemma 2.2 and the hypothesis, $V = \tilde{P} \subseteq \tilde{I} = aV \subseteq M$, which is absurd.

Conversely, assume that each nonzero nonmaximal prime ideal of V in not idempotent and let $I \subseteq J$ be ideals of V. If $I = \tilde{I}$, or $\tilde{J} = V$, then clearly $\tilde{I} \subseteq \tilde{J}$. If $\tilde{I} = V$, by Proposition 3.2, $I = P$ is an idempotent prime ideal of V. By the hypothesis, $I = M$. So $M = I \subseteq J \subseteq M$. Then $I = J = M$ and so $\tilde{I} = \tilde{J}$. Hence we may assume that $I \subsetneq \tilde{I} \subsetneq V$ and $\tilde{J} \subsetneq V$. By Proposition 3.2, $\tilde{I} = (IQ :_V Q)$, where $Q = II^{-1}$. Now, suppose that $\tilde{I} \not\subseteq \tilde{J}$. Then let $x \in \tilde{I} \setminus \tilde{J}$. Since V is a valuation domain, then $\tilde{J} \subsetneq xV$. So $x^{-1}I \subseteq x^{-1}J \subseteq x^{-1}\tilde{J} \subsetneq V \subseteq V_Q$. Since I is an ideal of $(I : I) = V_Q$, then $x^{-1}I \subseteq Q$. So $I \subseteq xQ \subseteq \tilde{I}Q = IQ \subseteq I$. Therefore $I = xQ$. If Q is nonmaximal, by the hypothesis, $Q^2 \subsetneq Q$. Hence $Q = aV_Q$ for some nonzero $a \in Q$ (since Q is the maximal ideal of V_Q). Hence $I = xQ = xaV_Q = xa(I : I)$. So I is stable and by Lemma 2.2, $\tilde{I} = I$, which is absurd. Hence $Q = M$ and $I = xM$. If M is principal in V, then so is I and therefore $\tilde{I} = I$, which is absurd. Hence $M = M^2$. So $\tilde{I} = (IM :_V M) = (xM^2 :_V M) = (xM :_V M) = x(M : M) = xV$. Let $b \in J \setminus I$. Then $xM = I \subsetneq bV$. Hence $xb^{-1}M \subseteq M$. So $xb^{-1} \in (M : M) = V$. Hence $x = (xb^{-1})b \in J \subseteq \tilde{J}$, which is absurd. It follows that $\tilde{I} \subseteq \tilde{J}$, as desired. $\square$

Now, we extend the Ratliff-Rush closure to arbitrary nonzero fractional ideals and we study its link to the notion of star operations. Our motivation is [12, Example 1.11], which provided an example of a Noetherian domain R with a nonzero ideal I such that $\widetilde{aI} \neq a\tilde{I}$ for some $0 \neq a \in R$. First, we recall that a star operation on R is a map $* : F(R) \longrightarrow F(R), E \mapsto E^*$, where $F(R)$ denotes the set of all nonzero fractional ideals of R, with the following properties for each $E, F \in F(R)$ and each $0 \neq a \in K$:
(E_1) $R^* = R$ and $(aE)^* = aE^*$;
(E_2) $E \subseteq E^*$ and if $E \subseteq F$, then $E^* \subseteq F^*$;
(E_3) $E^{**} = E^*$.

For more details on the notion of star operations, we refer the reader to [10].

Definition 3.4. Let R be an integral domain with quotient field K and let I be a nonzero fractional ideal of R.
(1) The generalized Ratliff-Rush closure of I is defined by $\hat{I} := \{x \in K | xI^n \subseteq I^{n+1}$, for some $n \geq 1\}$. Clearly $\tilde{I} = \hat{I} \cap R$ for any nonzero integral ideal I of R.

It is easy to see that for a nonzero fractional ideal I of a domain R, $\hat{I}$ is an R-module which is a fractional ideal if $(R : R^I) \neq 0$. In particular if R is conducive or L-stable, then $\hat{I}$ is always a fractional ideal of R. The next theorem gives necessary and sufficient conditions for the generalized Ratliff-Rush closure to be a star operation on a valuation domain.

Theorem 3.5. *Let V be a valuation domain. The generalized Ratllif-Rush closure on V is a star operation if and only if each nonzero nonmaximal prime ideal P of V is not idempotent. In this case, it coincides with the v-operation.*

Proof. Assume that the generalized Ratliff-Rush closure is a star operation. Then, by Proposition 3.3, each nonzero nonmaximal prime ideal of V is not idempotent. Conversely, assume that each nonzero nonmaximal prime ideal of V is not idempotent.
Claim. For each integral ideal I of V, $\tilde{I} = \hat{I}$. Indeed, it suffices to show that $\hat{I} \subseteq V$. If $II^{-1} = V$, then $\hat{I} = I$, as desired. Assume that $II^{-1} = Q$ is a prime ideal of V. Then $(I : I) = V_Q$. Let $x \in \hat{I}$. Then $xI^n \subseteq I^{n+1}$ for some positive integer n. Since $I^n I^{-n} = Q$, we get $xQ \subseteq IQ$. Now, if $Q = M$, then $xM \subseteq IM \subseteq M$. So $x \in (M : M) = V$. If $Q \subsetneq M$, by hypothesis, Q is not idempotent. Hence $Q = aV_Q$ (since Q is the maximal ideal of V_Q). So $xaV_Q \subseteq aIV_Q = aI$ (here I is an ideal of $(I : I) = V_Q$). Hence $xV_Q \subseteq I$ and therefore $x \in I \subseteq V$, as desired.
Now, we prove the three properties of star operations. Let I and J be nonzero fractional ideals of V and $o \neq a \in qf(V)$.
(1) (E_1): $x \in \widehat{aI}$ if and only if $x(aI)^n \subseteq (aI)^{n+1}$ for some positive integer n, if and only if $xa^{-1} \in (I^{n+1} : I^n) \subseteq \hat{I}$, if and only if $x \in a\hat{I}$.
(2) (E_2): Let $o \neq d \in V$ such that $dI \subseteq dJ \subseteq V$. By (E_1), Proposition 3.3(2) and the claim, $d\hat{I} = \widehat{dI} = \widetilde{dI} \subseteq \widetilde{dJ} = \widehat{dJ} = d\hat{J}$. Hence $\hat{I} \subseteq \hat{J}$.
(3) (E_3): Clearly $I \subseteq \hat{I}$ and by (E_1) and Proposition 3.3(1), $\hat{\hat{I}} = \hat{I}$.
To complete the proof, we prove that $\tilde{I} = I_v$ for each nonzero fractional ideal I of V. Since the v-operation is the largest star operation on V, then $\hat{I} \subseteq I_v$. Suppose that $\hat{I} \subsetneq I_v$ for some ideal I of V. Then I is not divisorial in V. Hence $I = aM$ for some $a \in qf(V)$ and $M = M^2$. Since M is idempotent, then M is not divisorial. So $M_v = V$. Hence $I_v = aM_v = aV = \hat{I}$ (note that by (E_1) and Lemma 2.2 $\hat{I} = a\hat{M} = a\tilde{M} = aV$), which is absurd. □

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